

Notes on noncommutative algebraic topology

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Abstract

A covariant functor from fibrations over the circle to a family of AF -algebras is constructed; the functor takes continuous maps between such fibrations to stable homomorphisms of the AF -algebras. We use this functor to develop an obstruction theory for the torus bundles of dimension 2, 3 and 4.

Key words and phrases: Anosov diffeomorphism, AF -algebra

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Introduction

Following Effros [5], noncommutative algebraic topology studies applications of the operator algebras in topology; such algebras are made of linear operators on a Hilbert space and, therefore, intrinsically noncommutative. One very fruitful approach to topology consists in construction of maps (functors) from the topological spaces to certain algebraic objects, so that continuous maps between the spaces become homomorphisms of the corresponding algebraic entities. The functors usually take value in the finitely generated groups (abelian or not) and, therefore, reduce topology to a simpler algebraic problem. The rings of operators on a Hilbert space are neither finitely generated nor commutative and, at the first glance, if ever such a reduction exists, it will never simplify the problem. Roughly speaking, our aim is to

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show, that this is wrong; we introduce an operator algebra, the so-called fundamental AF -algebra, which yields a set of simple obstructions (invariants) to existence of continuous maps in a class of manifolds fibering over the circle. One obstruction turns out to be the Galois group of the fundamental AF -algebra; this invariant dramatically simplifies for a class of the so-called tight torus bundles, so that topology boils down to a division test for a finite set of natural numbers.

A. AF -algebras ([4]). The C^* -algebra A is an algebra over the complex numbers \mathbb{C} with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*, a \in A$, such that A is complete with respect to the norm, and such that $\|ab\| \leq \|a\| \|b\|$ and $\|a^*a\| = \|a\|^2$ for every $a, b \in A$. Any commutative algebra A is isomorphic to the C^* -algebra $C_0(X)$ of continuous complex-valued functions on a locally compact Hausdorff space X ; the algebras which are not commutative are deemed as noncommutative topological spaces. A *stable homomorphism* $A \rightarrow A'$ is defined as the (usual) homomorphism $A \otimes \mathcal{K} \rightarrow A' \otimes \mathcal{K}$, where \mathcal{K} is the C^* -algebra of compact operators on a Hilbert space; such a homomorphism corresponds to a continuous map between the noncommutative spaces A and A' . The matrix algebra $M_n(\mathbb{C})$ is an example of noncommutative finite-dimensional C^* -algebra; a natural generalization are approximately finite-dimensional (AF -) algebras, which are given by an ascending sequence $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \dots$ of finite-dimensional semi-simple C^* -algebras $M_i = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$ and homomorphisms φ_i arranged into an infinite graph as follows. The two sets of vertices V_{i_1}, \dots, V_{i_k} and $V_{i'_1}, \dots, V_{i'_k}$ are joined by the b_{rs} edges, whenever the summand M_{i_r} contains b_{rs} copies of the summand $M_{i'_s}$ under the embedding φ_i ; as $i \rightarrow \infty$, one gets a *Bratteli diagram* of the AF -algebra. Such a diagram is defined by an infinite sequence of matrices $B_i = (b_{rs}^{(i)})$. If the homomorphisms $\varphi_1 = \varphi_2 = \dots = \text{Const}$, the AF -algebra is called *stationary*; its Bratteli diagram looks like an infinite graph with the incidence matrix $B = (b_{rs})$ repeated over and over again.

B. AF -algebra of measured foliation. Let M be a compact manifold of dimension m and \mathcal{F} a codimension k foliation of M with the holonomy, which preserves a measure (measured foliation) [8]; there exists a natural AF -algebra attached to \mathcal{F} as follows. Each measured foliation \mathcal{F} is given by a tangent plane $\omega(p) = 0$, where $\omega \in H^k(M; \mathbb{R})$ is a closed k -form and p a point of M ; let $\lambda_i > 0$ be periods of ω against a basis in the homology group $H_k(M)$. Consider the vector $\theta = (\theta_1, \dots, \theta_{n-1})$, where $\theta_i = \lambda_{i+1}/\lambda_1$ and $n = \text{rank } H_k(M)$. Let $\lim_{i \rightarrow \infty} B_i$ be the Jacobi-Perron continued frac-

tion convergent to the vector $(1, \theta)$; here $B_i \in GL_n(\mathbb{Z})$ are the non-negative matrices with $\det(B_i) = 1$ [3]. An AF -algebra $\mathbb{A}_{\mathcal{F}}$ is called *associated* to \mathcal{F} , if its Bratteli diagram is given by the matrices B_i . The algebra $\mathbb{A}_{\mathcal{F}}$ has a spate of remarkable properties, e.g. topologically conjugate (or, induced) foliations have stably isomorphic (or, stably homomorphic) AF -algebras (lemma 1); the dimension group of $\mathbb{A}_{\mathcal{F}}$ [5] coincides with the Plante group $P(\mathcal{F})$ of foliation \mathcal{F} [8].

C. Fundamental AF -algebras and main result. Let $\varphi : M \rightarrow M$ be an Anosov diffeomorphism of M [1]; if p is a fixed point of φ , then φ defines an invariant measured foliation \mathcal{F} of M given by the stable manifold $W^s(p)$ of φ at the point p [9], p.760. The associated AF -algebra $\mathbb{A}_{\mathcal{F}}$ is stationary (lemma 2); we call the latter a *fundamental AF -algebra* and denote it by $\mathbb{A}_{\varphi} := \mathbb{A}_{\mathcal{F}}$. Consider the mapping torus of φ , i.e. a manifold $M_{\varphi} := \{M \times [0, 1] \mid (x, 0) \sim (\varphi(x), 1), \forall x \in M\}$. Let \mathcal{M} be a category of the mapping tori of all Anosov's diffeomorphisms; the arrows of \mathcal{M} are continuous maps between the mapping tori. Likewise, let \mathcal{A} be a category of all fundamental AF -algebras; the arrows of \mathcal{A} are stable homomorphisms between the fundamental AF -algebras. By $F : \mathcal{M} \rightarrow \mathcal{A}$ we understand a map given by the formula $M_{\varphi} \mapsto \mathbb{A}_{\varphi}$, where $M_{\varphi} \in \mathcal{M}$ and $\mathbb{A}_{\varphi} \in \mathcal{A}$. Our main result can be stated as follows.

Theorem 1 *The map F is a functor, which sends each continuous map $N_{\psi} \rightarrow M_{\varphi}$ to a stable homomorphism $\mathbb{A}_{\psi} \rightarrow \mathbb{A}_{\varphi}$ of the corresponding fundamental AF -algebras.*

D. Applications. An application of theorem 1 is straightforward, since stable homomorphisms of the fundamental AF -algebras are easier to detect, than continuous maps between manifolds N_{ψ} and M_{φ} ; such homomorphisms are bijective with the inclusions of certain \mathbb{Z} -modules lying in a (real) algebraic number field. Often it is possible to prove, that no inclusion is possible and, thus, draw a topological conclusion about the maps (an obstruction theory). Namely, since \mathbb{A}_{ψ} is stationary, it has a constant incidence matrix B ; the splitting field of the polynomial $\det(B - xI)$ we denote by K_{ψ} and call $Gal(K_{\psi}|\mathbb{Q})$ a *Galois group* of the algebra \mathbb{A}_{ψ} . (The field K_{ψ} is always a Galois extension of \mathbb{Q} .) Suppose that $h : \mathbb{A}_{\psi} \rightarrow \mathbb{A}_{\varphi}$ is a stable homomorphism; since the corresponding invariant foliations \mathcal{F}_{ψ} and \mathcal{F}_{φ} are induced, their Plante groups are included $P(\mathcal{F}_{\varphi}) \subseteq P(\mathcal{F}_{\psi})$ and, therefore, $\mathbb{Q}(\lambda_{B'}) \subseteq K_{\psi}$, where $\lambda_{B'}$ is the Perron-Frobenius eigenvalue of the matrix B' attached to \mathbb{A}_{φ} . Thus, stable homomorphisms are bijective with subfields of the algebraic number

field K_ψ ; their classification achieves perfection in terms of the Galois theory, since the subfields are one-to-one with the subgroups of $Gal(\mathbb{A}_\psi)$ [7]. In particular, when $Gal(\mathbb{A}_\psi)$ is simple, there are only trivial stable homomorphisms; thus, the structure of $Gal(\mathbb{A}_\psi)$ is an obstruction (an invariant) to existence of a continuous map between the manifolds N_ψ and M_φ . Is our invariant effective? The answer is positive for a class of the so-called tight torus bundles; in this case N_ψ is given by a monodromy matrix, which is similar to the matrix B . The obstruction theory for the tight torus bundles of any dimension can be completely determined; it reduces to a divisibility test for a finite set of natural numbers. For the sake of clarity, the test is done in dimension $m = 2, 3$ and 4 and followed by the numerical examples.

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1 Preliminaries

1.1 Measured foliations

By a q -dimensional, class C^r foliation of an m -dimensional manifold M one understands a decomposition of M into a union of disjoint connected subsets

$\{\mathcal{L}_\alpha\}_{\alpha \in A}$, called the *leaves* [6]. They must satisfy the following property: each point in M has a neighborhood U and a system of local class C^r coordinates $x = (x^1, \dots, x^m) : U \rightarrow \mathbb{R}^m$ such that for each leaf \mathcal{L}_α , the components of $U \cap \mathcal{L}_\alpha$ are described by the equations $x^{q+1} = \text{Const}, \dots, x^m = \text{Const}$. Such a foliation is denoted by $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha \in A}$. The number $k = m - q$ is called a codimension of the foliation. An example of the codimension k foliation \mathcal{F} is given by a closed k -form ω on M : the leaves of \mathcal{F} are tangent to the plane $\omega(p) = 0$ at each point p of M . The C^r -foliations \mathcal{F}_0 and \mathcal{F}_1 of codimension k are said to be C^s -conjugate ($0 \leq s \leq r$), if there exists an (orientation-preserving) diffeomorphism of M , of class C^s , which maps the leaves of \mathcal{F}_0 onto the leaves of \mathcal{F}_1 ; when $s = 0$, \mathcal{F}_0 and \mathcal{F}_1 are *topologically conjugate*. Denote by $f : N \rightarrow M$ a map of class C^s ($1 \leq s \leq r$) of a manifold N into M ; the map f is said to be *transverse* to \mathcal{F} , if for all $x \in N$ it holds $T_y(M) = \tau_y(\mathcal{F}) + f_*T_x(N)$, where $\tau_y(\mathcal{F})$ are the vectors of $T_y(M)$ tangent to \mathcal{F} and $f_* : T_x(N) \rightarrow T_y(M)$ is the linear map on tangent vectors induced by f , where $y = f(x)$. If map $f : N \rightarrow M$ is transverse to a foliation $\mathcal{F}' = \{\mathcal{L}\}_{\alpha \in A}$ on M , then f induces a class C^s foliation \mathcal{F} on N , where the leaves are defined as $f^{-1}(\mathcal{L}_\alpha)$ for all $\alpha \in A$; it is immediate, that $\text{codim}(\mathcal{F}) = \text{codim}(\mathcal{F}')$. We shall call \mathcal{F} an *induced foliation*. When f is a submersion, it is transverse to any foliation of M ; in this case, the induced foliation \mathcal{F} is correctly defined for all \mathcal{F}' on M [6], p.373. Notice, that for $M = N$ foliations \mathcal{F} and \mathcal{F}' are topologically conjugate. To introduce measured foliations, denote by P and Q two k -dimensional submanifolds of M , which are everywhere transverse to a foliation \mathcal{F} of codimension k . Consider a collection of C^r homeomorphisms between subsets of P and Q determined by restricting deformations of M , which preserve each leaf of \mathcal{F} ; in other words, the homeomorphisms are induced by a return map along the leaves of \mathcal{F} . The collection of all such homeomorphisms between subsets of all possible pairs of transverse manifolds generates a *holonomy pseudogroup* of \mathcal{F} under composition of the homeomorphisms [8], p.329. A foliation \mathcal{F} is said to have measure preserving holonomy, if its holonomy pseudogroup has a non-trivial invariant measure, which is finite on compact sets; for brevity, we call \mathcal{F} a *measured foliation*. An example of measured foliation is a foliation, determined by closed k -form ω ; the restriction of ω to a transverse k -dimensional manifold determines a volume element, which gives a positive invariant measure on open sets. Each measured foliation \mathcal{F} defines an element of the cohomology group $H^k(M; \mathbb{R})$ [8]; in the case of \mathcal{F} given by a closed k -form ω , such an element coincides with the de Rham cohomology

class of ω , *ibid.* In view of the isomorphism $H^k(M; \mathbb{R}) \cong \text{Hom}(H_k(M), \mathbb{R})$, foliation \mathcal{F} defines a linear map h from the k -th homology group $H_k(M)$ to \mathbb{R} ; by the *Plante group* $P(\mathcal{F})$ we shall understand a finitely generated abelian subgroup $h(H_k(M)/\text{Tors})$ of the real line \mathbb{R} . If $\{\gamma_i\}$ is a basis of the homology group $H_k(M)$, then the periods $\lambda_i = \int_{\gamma_i} \omega$ are generators of the group $P(\mathcal{F})$ [8].

1.2 AF -algebra of measured foliation

Let $(\lambda_1, \dots, \lambda_n)$ be a basis of the Plante group $P(\mathcal{F})$ of measured foliation \mathcal{F} , such that $\lambda_i > 0$. Take a vector $\theta = (\theta_1, \dots, \theta_{n-1})$ with $\theta_i = \lambda_{i+1}/\lambda_1$; the Jacobi-Perron continued fraction of θ is given by the formula [3]:

$$\left(\begin{smallmatrix} 1 \\ \theta \end{smallmatrix} \right) = \lim_{i \rightarrow \infty} \left(\begin{smallmatrix} 0 & 1 \\ I & b_1 \end{smallmatrix} \right) \cdots \left(\begin{smallmatrix} 0 & 1 \\ I & b_i \end{smallmatrix} \right) \left(\begin{smallmatrix} 0 \\ \mathbb{I} \end{smallmatrix} \right) = \lim_{i \rightarrow \infty} B_i \left(\begin{smallmatrix} 0 \\ \mathbb{I} \end{smallmatrix} \right), \quad (1)$$

where $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^T$ is a vector of the non-negative integers, I the unit matrix and $\mathbb{I} = (0, \dots, 0, 1)^T$. An AF -algebra given by the Bratteli diagram with the incidence matrices B_i will be called *associated* to foliation \mathcal{F} ; we shall denote such an algebra by $\mathbb{A}_{\mathcal{F}}$. Note, that if \mathcal{F}' is a measured foliation on a manifold M and $f : N \rightarrow M$ is a submersion, the induced foliation \mathcal{F} on N is a measured foliation. We shall denote by \mathcal{MFol} a category of all manifolds with measured foliations (of fixed codimension), whose arrows are submersions of the manifolds; by $\mathcal{M}_0\mathcal{Fol}$ we understand a subcategory of \mathcal{MFol} , consisting of manifolds, whose foliations have a unique transverse measure. Let \mathcal{Rng} be a category of the AF -algebras given by *convergent* Jacobi-Perron fractions (1), so that the arrows of \mathcal{Rng} are the stable homomorphisms of the AF -algebras. By F we denote a map between $\mathcal{M}_0\mathcal{Fol}$ and \mathcal{Rng} given by the formula $\mathcal{F} \mapsto \mathbb{A}_{\mathcal{F}}$. Notice, that F is correctly defined, since foliations with the unique measure unfold into a convergent Jacobi-Perron fraction; this assertion follows from [2].

Lemma 1 *The map $F : \mathcal{M}_0\mathcal{Fol} \longrightarrow \mathcal{Rng}$ is a functor, which sends any pair of induced foliations to a pair of stably homomorphic AF -algebras.*

1.3 Fundamental AF -algebras

Let M be an m -dimensional manifold and $\varphi : M \rightarrow M$ a diffeomorphism of M ; recall, that an orbit of point $x \in M$ is the subset $\{\varphi^n(x) \mid n \in \mathbb{Z}\}$ of M .

The finite orbits $\varphi^m(x) = x$ are called periodic; when $m = 1$, x is a *fixed point* of diffeomorphism φ . The fixed point p is *hyperbolic* if the eigenvalues λ_i of the linear map $D\varphi(p) : T_p(M) \rightarrow T_p(M)$ do not lie at the unit circle. If $p \in M$ is a hyperbolic fixed point of a diffeomorphism $\varphi : M \rightarrow M$, denote by $T_p(M) = V^s + V^u$ the corresponding decomposition of the tangent space under the linear map $D\varphi(p)$, where V^s (V^u) is the eigenspace of $D\varphi(p)$ corresponding to $|\lambda_i| > 1$ ($|\lambda_i| < 1$). There exists a contraction $g : W^s(p) \rightarrow W^s(p)$ with fixed point p_0 and an injective equivariant immersion $J : W^s(p) \rightarrow M$, such that $J(p_0) = p$ and $DJ(p_0) : T_{p_0}(W^s(p)) \rightarrow T_p(M)$ is an isomorphism; the image of J defines an immersed submanifold $W^s(p) \subset M$ called a *stable manifold* of φ at p . Clearly, $\dim(W^s(p)) = \dim(V^s)$. We say that $\varphi : M \rightarrow M$ is an *Anosov diffeomorphism* ([1]) if the following condition is satisfied: there exists a splitting of the tangent bundle $T(M)$ into a continuous Whitney sum $T(M) = E^s + E^u$, invariant under $D\varphi : T(M) \rightarrow T(M)$, so that $D\varphi : E^s \rightarrow E^s$ is contracting and $D\varphi : E^u \rightarrow E^u$ is expanding¹. Let p be a fixed point of the Anosov diffeomorphism $\varphi : M \rightarrow M$ and $W^s(p)$ its stable manifold. Since $W^s(p) \rightarrow M$ is a dense immersion, the closure of $W^s(p)$ is the entire M ; moreover, if q is a periodic point of φ of period n , then $W^s(q)$ is a translate of $W^s(p)$, i.e. $\varphi^n(W^s(q)) = W^s(p)$. Consider a foliation \mathcal{F} of M , whose leaves are the translates of $W^s(p)$; the \mathcal{F} is a continuous foliation ([9], p.760), which is invariant under the action of diffeomorphism φ on its leaves, i.e. φ moves leaves of \mathcal{F} to the leaves of \mathcal{F} . The holonomy of \mathcal{F} preserves the Lebesgue measure and, therefore, \mathcal{F} is a measured foliation; we shall call it an *invariant measured foliation* and denote by \mathcal{F}_φ . The *AF*-algebra of foliation \mathcal{F} is called *fundamental*, when $\mathcal{F} = \mathcal{F}_\varphi$, where φ is an Anosov diffeomorphism; the following is a basic property of such algebras.

Lemma 2 *Any fundamental AF-algebra is stationary.*

¹ It follows from definition, that the Anosov diffeomorphism imposes a restriction on topology of manifold M , in the sense that not each manifold can support such a diffeomorphism; however, if one Anosov diffeomorphism exists on M , there are infinitely many (conjugacy classes of) such diffeomorphisms on M . It is an open problem of S. Smale, which M can carry an Anosov diffeomorphism; so far, it is proved that the hyperbolic diffeomorphisms of m -dimensional tori and certain automorphisms of the nilmanifolds are Anosov's [9]. It is worth mentioning, that on each two-dimensional manifold (a surface of genus $g \geq 1$) there exists a rich family of the so-called pseudo-Anosov diffeomorphisms [10], to which our theory fully applies.

2 Proofs

2.1 Proof of lemma 1

Let \mathcal{F}' be measured foliation on M , given by a closed form $\omega' \in H^k(M; \mathbb{R})$; let \mathcal{F} be measured foliation on N , induced by a submersion $f : N \rightarrow M$. Roughly speaking, we have to prove, that diagram in Fig.1 is commutative; the proof amounts to the fact, that the periods of form ω' are contained among the periods of form $\omega \in H^k(N; \mathbb{R})$ corresponding to the foliation \mathcal{F} . The map f defines a homomorphism $f_* : H_k(N) \rightarrow H_k(M)$ of the k -

$$\begin{array}{ccc}
 \mathcal{F} & \xrightarrow{\text{induction}} & \mathcal{F}' \\
 \downarrow & & \downarrow \\
 \mathbb{A}_{\mathcal{F}} & \xrightarrow[\text{homomorphism}]{\text{stable}} & \mathbb{A}_{\mathcal{F}'}
 \end{array}$$

Figure 1: Functor $F : \mathcal{M}_0\mathcal{Fol} \longrightarrow \mathcal{Rng}$.

th homology groups; let $\{e_i\}$ and $\{e'_i\}$ be a basis in $H_k(N)$ and $H_k(M)$, respectively. Since $H_k(M) = H_k(N) / \ker(f_*)$, we shall denote by $[e_i] := e_i + \ker(f_*)$ a coset representative of e_i ; these can be identified with the elements $e_i \notin \ker(f_*)$. The integral $\int_{e_i} \omega$ defines a scalar product $H_k(N) \times H^k(N; \mathbb{R}) \rightarrow \mathbb{R}$, so that f_* is a linear self-adjoint operator; thus, we can write:

$$\lambda'_i = \int_{e'_i} \omega' = \int_{e'_i} f^*(\omega) = \int_{f_*^{-1}(e'_i)} \omega = \int_{[e_i]} \omega \in P(\mathcal{F}), \quad (2)$$

where $P(\mathcal{F})$ is the Plante group (the group of periods) of foliation \mathcal{F} . Since λ'_i are generators of $P(\mathcal{F}')$, we conclude that $P(\mathcal{F}') \subseteq P(\mathcal{F})$. Note, that $P(\mathcal{F}') = P(\mathcal{F})$ if and only if f_* is an isomorphism.

One can apply a criterion of the stable homomorphism of AF -algebras; namely, $\mathbb{A}_{\mathcal{F}}$ and $\mathbb{A}_{\mathcal{F}'}$ are stably homomorphic, if and only if, there exists a positive homomorphism $h : G \rightarrow H$ between their dimension groups G and H [5], p.15. But $G \cong P(\mathcal{F})$ and $H \cong P(\mathcal{F}')$, while $h = f_*$. Lemma 1 follows.

□

2.2 Proof of lemma 2

Let $\varphi : M \rightarrow M$ be an Anosov diffeomorphism; we proceed by showing, that invariant foliation \mathcal{F}_φ is given by form $\omega \in H^k(M; \mathbb{R})$, which is an eigenvector of the linear map $[\varphi] : H^k(M; \mathbb{R}) \rightarrow H^k(M; \mathbb{R})$ induced by φ . Indeed, let $0 < c < 1$ be contracting constant of the stable sub-bundle E^s of diffeomorphism φ and Ω the corresponding volume element; by definition, $\varphi(\Omega) = c\Omega$. Note, that Ω is given by restriction of form ω to a k -dimensional manifold, transverse to the leaves of \mathcal{F}_φ . The leaves of \mathcal{F}_φ are fixed by φ and, therefore, $\varphi(\Omega)$ is given by a multiple $c\omega$ of form ω . Since $\omega \in H^k(M; \mathbb{R})$ is a vector, whose coordinates define \mathcal{F}_φ up to a scalar, we conclude, that $[\varphi](\omega) = c\omega$, i.e. ω is an eigenvector of the linear map $[\varphi]$. Let $(\lambda_1, \dots, \lambda_n)$ be a basis of the Plante group $P(\mathcal{F}_\varphi)$, such that $\lambda_i > 0$. Notice, that φ acts on λ_i as multiplication by constant c ; indeed, since $\lambda_i = \int_{\gamma_i} \omega$, we have:

$$\lambda'_i = \int_{\gamma_i} [\varphi](\omega) = \int_{\gamma_i} c\omega = c \int_{\gamma_i} \omega = c\lambda_i, \quad (3)$$

where $\{\gamma_i\}$ is a basis in $H_k(M)$. Since φ preserves the leaves of \mathcal{F}_φ , one concludes that $\lambda'_i \in P(\mathcal{F}_\varphi)$; therefore, $\lambda'_j = \sum b_{ij} \lambda_i$ for a non-negative integer matrix $B = (b_{ij})$. According to [2], matrix B can be written as a finite product:

$$B = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_p \end{pmatrix} := B_1 \dots B_p, \quad (4)$$

where $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^T$ is a vector of non-negative integers and I the unit matrix. Let $\lambda = (\lambda_1, \dots, \lambda_n)$. Consider a purely periodic Jacobi-Perron continued fraction:

$$\lim_{i \rightarrow \infty} \overline{B_1 \dots B_p} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \quad (5)$$

where $\mathbb{I} = (0, \dots, 0, 1)^T$; by a basic property of such fractions, it converges to an eigenvector $\lambda' = (\lambda'_1, \dots, \lambda'_n)$ of matrix $B_1 \dots B_p$ [3], Ch.3. But $B_1 \dots B_p = B$ and λ is an eigenvector of matrix B ; therefore, vectors λ and λ' are collinear. The collinear vectors are known to have the same continued fractions; thus, we have

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{i \rightarrow \infty} \overline{B_1 \dots B_p} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \quad (6)$$

where $\theta = (\theta_1, \dots, \theta_{n-1})$ and $\theta_i = \lambda_{i+1}/\lambda_1$. Since vector $(1, \theta)$ unfolds into a periodic Jacobi-Perron continued fraction, we conclude, that the AF -algebra \mathbb{A}_φ is stationary. Lemma 2 is proved. \square

2.3 Proof of theorem 1

Let $\psi : N \rightarrow N$ and $\varphi : M \rightarrow M$ be a pair of Anosov diffeomorphisms; denote by (N, \mathcal{F}_ψ) and (M, \mathcal{F}_φ) the corresponding invariant foliations of manifolds N and M , respectively. In view of lemma 1, it is sufficient to prove, that the diagram in Fig.2 is commutative. We shall split the proof in a series of lemmas.

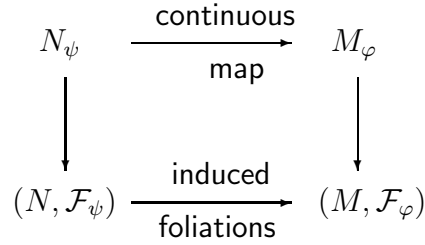


Figure 2: Mapping tori and invariant foliations.

Lemma 3 *There exists a continuous map $N_\psi \rightarrow M_\varphi$, whenever $f \circ \varphi = \psi \circ f$ for a submersion $f : N \rightarrow M$.*

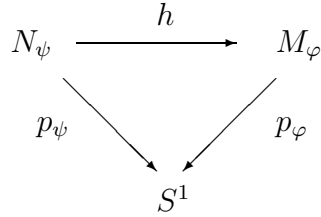


Figure 3: The fiber bundles N_ψ and M_φ over S^1 .

Proof. (i) Suppose, that $h : N_\psi \rightarrow M_\varphi$ is a continuous map; let us show, that there exists a submersion $f : N \rightarrow M$, such that $f \circ \varphi = \psi \circ f$. Both N_ψ and M_φ fiber over the circle S^1 with the projection map p_ψ and p_φ , respectively; therefore, the diagram in Fig.3 is commutative. Let $x \in S^1$; since $p_\psi^{-1} = N$ and $p_\varphi^{-1} = M$, the restriction of h to x defines a submersion $f : N \rightarrow M$,

i.e. $f = h_x$. Moreover, since ψ and φ are monodromy maps of the bundle, it holds:

$$\begin{cases} p_\psi^{-1}(x + 2\pi) &= \psi(N), \\ p_\varphi^{-1}(x + 2\pi) &= \varphi(M). \end{cases} \quad (7)$$

From the diagram in Fig.3, we get: $\psi(N) = p_\psi^{-1}(x + 2\pi) = f^{-1}(p_\varphi^{-1}(x + 2\pi)) = f^{-1}(\varphi(M)) = f^{-1}(\varphi(f(N)))$; thus, $f \circ \psi = \varphi \circ f$. The necessary condition of lemma 3 follows.

(ii) Suppose, that $f : N \rightarrow M$ is a submersion, such that $f \circ \varphi = \psi \circ f$; we have to construct a continuous map $h : N_\psi \rightarrow M_\varphi$. Recall, that

$$\begin{cases} N_\psi &= \{N \times [0, 1] \mid (x, 0) \sim (\psi(x), 1)\}, \\ M_\varphi &= \{M \times [0, 1] \mid (y, 0) \sim (\varphi(y), 1)\}. \end{cases} \quad (8)$$

We shall identify the points of N_ψ and M_φ using the substitution $y = f(x)$; it remains to verify, that such an identification will satisfy the gluing condition $y \sim \varphi(y)$. In view of condition $f \circ \varphi = \psi \circ f$, we have:

$$y = f(x) \sim f(\psi(x)) = \varphi(f(x)) = \varphi(y). \quad (9)$$

Thus, $y \sim \varphi(y)$ and, therefore, the map $h : N_\psi \rightarrow M_\varphi$ is continuous. The sufficient condition of lemma 3 is proved. \square

$$\begin{array}{ccc} H^k(N; \mathbb{R}) & \xrightarrow{[\psi]} & H^k(N, \mathbb{R}) \\ [f] \downarrow & & \downarrow [f] \\ H^k(M, \mathbb{R}) & \xrightarrow{[\varphi]} & H^k(M, \mathbb{R}) \end{array}$$

Figure 4: The linear maps $[\psi]$, $[\varphi]$ and $[f]$.

Lemma 4 *If a submersion $f : N \rightarrow M$ satisfies condition $f \circ \varphi = \psi \circ f$ for the Anosov diffeomorphisms $\psi : N \rightarrow N$ and $\varphi : M \rightarrow M$, then the invariant foliations (N, \mathcal{F}_ψ) and (M, \mathcal{F}_φ) are induced by f .*

Proof. The invariant foliations \mathcal{F}_ψ and \mathcal{F}_φ are measured; we shall denote by $\omega_\psi \in H^k(N; \mathbb{R})$ and $\omega_\varphi \in H^k(M; \mathbb{R})$ the corresponding cohomology class, respectively. The linear maps on $H^k(N; \mathbb{R})$ and $H^k(M; \mathbb{R})$ induced by ψ and φ , we shall denote by $[\psi]$ and $[\varphi]$; the linear map between $H^k(N; \mathbb{R})$ and $H^k(M; \mathbb{R})$ induced by f , we write as $[f]$. Notice, that $[\psi]$ and $[\varphi]$ are isomorphisms, while $[f]$ is generally a homomorphism. It was shown earlier, that ω_ψ and ω_φ are eigenvectors of linear maps $[\psi]$ and $[\varphi]$, respectively; in other words, we have:

$$\begin{cases} [\psi]\omega_\psi &= c_1\omega_\psi, \\ [\varphi]\omega_\varphi &= c_2\omega_\varphi, \end{cases} \quad (10)$$

where $0 < c_1 < 1$ and $0 < c_2 < 1$. Consider a diagram in Fig.4, which involves the linear maps $[\psi]$, $[\varphi]$ and $[f]$; the diagram is commutative, since condition $f \circ \varphi = \psi \circ f$ implies, that $[\varphi] \circ [f] = [f] \circ [\psi]$. Take the eigenvector ω_ψ and consider its image under the linear map $[\varphi] \circ [f]$:

$$[\varphi] \circ [f](\omega_\psi) = [f] \circ [\psi](\omega_\psi) = [f](c_1\omega_\psi) = c_1 ([f](\omega_\psi)). \quad (11)$$

Therefore, vector $[f](\omega_\psi)$ is an eigenvector of the linear map $[\varphi]$; let compare it with the eigenvector ω_φ :

$$\begin{cases} [\varphi]([f](\omega_\psi)) &= c_1 ([f](\omega_\psi)), \\ [\varphi]\omega_\varphi &= c_2\omega_\varphi. \end{cases} \quad (12)$$

We conclude, therefore, that ω_φ and $[f](\omega_\psi)$ are collinear vectors, such that $c_1^m = c_2^n$ for some integers $m, n > 0$; a scaling gives us $[f](\omega_\psi) = \omega_\varphi$. The latter is an analytic formula, which says that the submersion $f : N \rightarrow M$ induces foliation (N, \mathcal{F}_ψ) from the foliation (M, \mathcal{F}_φ) . Lemma 4 is proved. \square

To finish our proof of theorem 1, let $N_\psi \rightarrow M_\varphi$ be a continuous map; by lemma 3, there exists a submersion $f : N \rightarrow M$, such that $f \circ \varphi = \psi \circ f$. Lemma 4 says, that in this case the invariant measured foliations (N, \mathcal{F}_ψ) and (M, \mathcal{F}_φ) are induced. On the other hand, from lemma 2 we know, that the Jacobi-Perron continued fraction connected to foliations \mathcal{F}_ψ and \mathcal{F}_φ are periodic and, hence, convergent [3]; therefore, one can apply lemma 1, which says that the AF -algebra \mathbb{A}_ψ is stably homomorphic to the AF -algebra \mathbb{A}_φ . The latter are, by definition, the fundamental AF -algebras of the Anosov diffeomorphisms ψ and φ , respectively. Theorem 1 is proved. \square

3 Applications of theorem 1

3.1 Galois group of the fundamental AF -algebra

Any fundamental AF -algebra \mathbb{A}_ψ is given by a single positive integer matrix B (lemma 2); we shall denote by K_ψ the splitting field of the characteristic polynomial of B . Since B is positive, $\text{char}(B)$ is irreducible; indeed, if $\text{char}(B)$ were reducible, B can be written in a block diagonal form, none of whose power is positive. Therefore, K_ψ is a Galois extension of \mathbb{Q} . We call $\text{Gal}(\mathbb{A}_\psi) := \text{Gal}(K_\psi|\mathbb{Q})$ a *Galois group* of the algebra \mathbb{A}_ψ . The second algebraic field is connected to the Perron-Frobenius eigenvalue λ_B of the matrix B ; we shall denote this field $\mathbb{Q}(\lambda_B)$. Note, that $\mathbb{Q}(\lambda_B) \subseteq K_\psi$ and $\mathbb{Q}(\lambda_B)$ is not, in general, a Galois extension of \mathbb{Q} ; the reason being complex roots the polynomial $\text{char}(B)$ may have and if there are no such roots $\mathbb{Q}(\lambda_B) = K_\psi$. There is still a group $\text{Aut}(\mathbb{Q}(\lambda_B))$ of automorphisms of $\mathbb{Q}(\lambda_B)$ fixing the field \mathbb{Q} and $\text{Aut}(\mathbb{Q}(\lambda_B)) \subseteq \text{Gal}(K_\psi)$ is a subgroup inclusion.

Lemma 5 *If $h : \mathbb{A}_\psi \rightarrow \mathbb{A}_\varphi$ is a stable homomorphism, then $\mathbb{Q}(\lambda_{B'}) \subseteq K_\psi$ is a field inclusion.*

Proof. Notice, that the non-negative matrix B becomes strictly positive, when a proper power of it is taken; we always assume B positive. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a basis of the Plante group $P(\mathcal{F}_\psi)$. Following the proof of lemma 2, one concludes that $\lambda_i \in K_\psi$; indeed, $\lambda_B \in K_\psi$ is the Perron-Frobenius eigenvalue of B , while λ the corresponding eigenvector. The latter can be scaled so, that $\lambda_i \in K_\psi$. Any stable homomorphism $h : \mathbb{A}_\psi \rightarrow \mathbb{A}_\varphi$ induces a positive homomorphism of their dimension groups $[h] : G \rightarrow H$; but $G \cong P(\mathcal{F}_\psi)$ and $H \cong P(\mathcal{F}_\varphi)$. From inclusion $P(\mathcal{F}_\varphi) \subseteq P(\mathcal{F}_\psi)$, one gets $\mathbb{Q}(\lambda_{B'}) \cong P(\mathcal{F}_\varphi) \otimes \mathbb{Q} \subseteq P(\mathcal{F}_\psi) \otimes \mathbb{Q} \cong \mathbb{Q}(\lambda_B) \subseteq K_\psi$ and, therefore, $\mathbb{Q}(\lambda_{B'}) \subseteq K_\psi$. Lemma 5 follows. \square

Corollary 1 *If $h : \mathbb{A}_\psi \rightarrow \mathbb{A}_\varphi$ is a stable homomorphism, then $\text{Aut}(\mathbb{Q}(\lambda_{B'}))$ (or, $\text{Gal}(\mathbb{A}_\varphi)$) is a subgroup (or, a normal subgroup) of $\text{Gal}(\mathbb{A}_\psi)$.*

Proof. The (Galois) subfields of the Galois field K_ψ are bijective with the (normal) subgroups of the group $\text{Gal}(K_\psi)$ [7]. \square

3.2 Tight torus bundles

Let $T^m \cong \mathbb{R}^m/\mathbb{Z}^m$ be an m -dimensional torus; consider ψ_0 a $m \times m$ matrix with integer entries and determinant ± 1 . Then ψ_0 can be thought of as a

linear transformation of \mathbb{R}^m , which preserves the lattice $L \cong \mathbb{Z}^m$ of points with integer coordinates. There is an induced diffeomorphism ψ of the quotient $T^m \cong \mathbb{R}^m/\mathbb{Z}^m$ onto itself; this diffeomorphism $\psi : T^m \rightarrow T^m$ has a fixed point p corresponding to the origin of \mathbb{R}^m . Suppose that ψ_0 is hyperbolic, i.e. there are no eigenvalues of ψ_0 at the unit circle; then p is a hyperbolic fixed point of ψ and the stable manifold $W^s(p)$ is the image of the corresponding eigenspace of ψ_0 under the projection $\mathbb{R}^m \rightarrow T^m$. We shall say, that ψ_0 is *tight* if $\text{codim } W^s(p) = 1$; in other words, the eigenvalue of ψ_0 inside (or outside) the unit circle is unique. The hyperbolic automorphisms of two and three-dimensional tori are always tight.

Lemma 6 *When ψ_0 is tight, it is similar to the matrix B defined by the algebra \mathbb{A}_ψ .*

Proof. Since $H_k(T^m; \mathbb{R}) \cong \mathbb{R}^{\frac{m!}{k!(m-k)!}}$, one gets $H_{m-1}(T^m; \mathbb{R}) \cong \mathbb{R}^m$; in view of the Poincaré duality, $H^1(T^m; \mathbb{R}) = H_{m-1}(T^m; \mathbb{R}) \cong \mathbb{R}^m$. Since $\text{codim } W^s(p) = 1$, measured foliation \mathcal{F}_ψ is given by a closed form $\omega_\psi \in H^1(T^m; \mathbb{R})$, such that $[\psi]\omega_\psi = \lambda_\psi \omega_\psi$, where λ_ψ is the eigenvalue of the linear transformation $[\psi] : H^1(T^m; \mathbb{R}) \rightarrow H^1(T^m; \mathbb{R})$. But $[\psi] = \psi_0$, because $H^1(T^m; \mathbb{R})$ is the universal cover for T^m ; the eigenspace $W^u(p)$ of ψ_0 corresponds to the eigenform ω_ψ . Recall, that the Plante group $P(\mathcal{F}_\psi)$ is generated by coordinates of vector $\omega_\psi \in H^1(T^m; \mathbb{R})$; matrix B corresponds to ψ_0 , written in a basis $\lambda = (\lambda_1, \dots, \lambda_m)$, such that $B\lambda = c\lambda$, where $\lambda_i > 0$, see proof of lemma 2. But any change of basis in the \mathbb{Z} -module $P(\mathcal{F}_\psi)$ is given by an integer invertible matrix S ; therefore, $B = S^{-1}\psi_0 S$. Lemma 6 follows. \square

Let $\psi : T^m \rightarrow T^m$ be a hyperbolic diffeomorphism; the mapping torus T_ψ^m will be called a (hyperbolic) *torus bundle* of dimension m . Let $k = |\text{Gal}(\mathbb{A}_\psi)|$; it follows from the Galois theory, that $1 < k \leq m!$. Denote t_i the cardinality of a subgroup $G_i \subseteq \text{Gal}(\mathbb{A}_\psi)$.

Corollary 2 *There are no continuous map $T_\psi^m \rightarrow T_\varphi^{m'}$, whenever $t'_i \nmid k$ for all $G'_i \subseteq \text{Gal}(\mathbb{A}_\varphi)$.*

Proof. If $h : T_\psi^m \rightarrow T_\varphi^{m'}$ was a continuous map to a torus bundle of dimension $m' < m$, then, by theorem 1 and corollary 1, the $\text{Aut}(\mathbb{Q}(\lambda_{B'}))$ (or, $\text{Gal}(\mathbb{A}_\varphi)$) were a non-trivial subgroup (or, normal subgroup) of the group $\text{Gal}(\mathbb{A}_\psi)$; since $k = |\text{Gal}(\mathbb{A}_\psi)|$, one concludes that one of t'_i divides k . This contradicts our assumption. \square

Definition 1 *The torus bundle T_ψ^m is called robust, if there exists $m' < m$, such that no continuous map $T_\psi^m \rightarrow T_\varphi^{m'}$ is possible.*

Are there robust bundles? It is shown in this section, that for $m = 2, 3$ and 4 there are infinitely many robust torus bundles; a reasonable guess is that it is true in any dimension.

3.2.1 Case $m = 2$

This case is trivial; ψ_0 is a hyperbolic matrix and always tight. The $\text{char}(\psi_0) = \text{char}(B)$ is an irreducible quadratic polynomial with two real roots; $\text{Gal}(\mathbb{A}_\psi) \cong \mathbb{Z}_2$ and, therefore, $|\text{Gal}(\mathbb{A}_\psi)| = 2$. Formally, T_ψ^2 is robust, since no torus bundle of a smaller dimension is defined.

3.2.2 Case $m = 3$

The ψ_0 is hyperbolic; it is always tight, since one root of $\text{char}(\psi_0)$ is real and isolated inside or outside the unit circle.

Corollary 3 *Let*

$$\psi_0(b, c) = \begin{pmatrix} -b & 1 & 0 \\ -c & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad (13)$$

be such, that $\text{char}(\psi_0(b, c)) = x^3 + bx^2 + cx + 1$ is irreducible and $-4b^3 + b^2c^2 + 18bc - 4c^3 - 27$ is the square of an integer; then T_ψ^3 admits no continuous map to any T_φ^2 .

Proof. The $\text{char}(\psi_0(b, c)) = x^3 + bx^2 + cx + 1$ and the discriminant $D = -4b^3 + b^2c^2 + 18bc - 4c^3 - 27$. By Theorem 13.1 [7], $\text{Gal}(\mathbb{A}_\psi) \cong \mathbb{Z}_3$ and, therefore, $k = |\text{Gal}(\mathbb{A}_\psi)| = 3$. For $m' = 2$, it was shown that $\text{Gal}(\mathbb{A}_\varphi) \cong \mathbb{Z}_2$ and, therefore, $t'_1 = 2$. Since $2 \nmid 3$, corollary 2 says that no continuous map $T_\psi^3 \rightarrow T_\varphi^2$ can be constructed. \square

Example 1. There are infinitely many matrices $\psi_0(b, c)$ satisfying the assumptions of corollary 3; below are a few numerical examples of robust bundles:

$$\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Notice, that the above matrices are not pairwise similar; it can be gleaned from their traces. Thus, they represent topologically distinct torus bundles.

3.2.3 Case $m = 4$

Let $p(x) = x^4 + ax^3 + bx^2 + cx + d$ be a quartic. Consider the associated cubic polynomial $r(x) = x^3 - bx^2 + (ac - 4d)x + 4bd - a^2d - c^2$; denote by L the splitting field of $r(x)$.

Corollary 4 *Let*

$$\psi_0(a, b, c) = \begin{pmatrix} -a & 1 & 0 & 0 \\ -b & 0 & 1 & 0 \\ -c & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad (14)$$

be tight and such, that $\text{char}(\psi_0(a, b, c)) = x^4 + ax^3 + bx^2 + cx + 1$ is irreducible and one of the following holds: (i) $L = \mathbb{Q}$; (ii) $r(x)$ has a unique root $t \in \mathbb{Q}$ and $h(x) = (x^2 - tx + 1)[x^2 + ax + (b - t)]$ splits over L ; (iii) $r(x)$ has a unique root $t \in \mathbb{Q}$ and $h(x)$ does not split over L . Then T_ψ^4 admits no continuous map to any T_φ^3 with $D > 0$.

Proof. According to Theorem 13.4 [7], $\text{Gal}(\mathbb{A}_\psi) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in case (i); $\text{Gal}(\mathbb{A}_\psi) \cong \mathbb{Z}_4$ in case (ii); and $\text{Gal}(\mathbb{A}_\psi) \cong D_4$ (the dihedral group) in case (iii). Therefore, $k = |\mathbb{Z}_2 \oplus \mathbb{Z}_2| = |\mathbb{Z}_4| = 4$ or $k = |D_4| = 8$. On the other hand, for $m' = 3$ with $D > 0$ (all roots are real), we have $t'_1 = |\mathbb{Z}_3| = 3$ and $t'_2 = |S_3| = 6$. Since $3; 6 \nmid 4; 8$, corollary 2 says that continuous map $T_\psi^4 \rightarrow T_\varphi^3$ is impossible. \square

Example 2. There are infinitely many matrices ψ_0 , which satisfy the assumption of corollary 4; indeed, consider a family

$$\psi_0(a, c) = \begin{pmatrix} -2a & 1 & 0 & 0 \\ -a^2 - c^2 & 0 & 1 & 0 \\ -2c & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (15)$$

where $a, c \in \mathbb{Z}$. The associated cubic becomes $r(x) = x[x^2 - (a^2 + c^2)x + 4(ac - 1)]$, so that $t = 0$ is a rational root; then $h(x) = (x^2 + 1)[x^2 + 2ax + a^2 + c^2]$. The matrix $\psi_0(a, c)$ satisfies one of the conditions (i)-(iii) of corollary 4 for each $a, c \in \mathbb{Z}$; it remains to eliminate the (non-generic) matrices, which are not tight or irreducible. Thus, $\psi_0(a, c)$ defines a family of topologically distinct robust bundles.

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